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Abstract. The classical 'Kirchhoff's theorem' (the energy density of the radiation at equilibrium at high temperature, T, is a function of T only) is used to obtain the Casimir energy at zero temperature without recourse to regularization. The validity of 'Kirchhoff's theorem' at the high-temperature limit for the case at hand is confirmed. The Casimir entropy is defined and its temperature dependence is displayed. The Casimir entropy at high temperatures is shown to approach a positive *geometry-dependent* but *temperature-independent* constant.

1. Introduction

The zero-point energy of vacuum fluctuations is an important result in the theory of quantized fields [1]. It is of interest in various branches of physics starting from liquid helium and extending all the way to the cosmological constant [2]. The Casimir effect deals with the modification of this energy (or energies when we consider not only the electromagnetic field) due to constraining boundaries. The original analysis [3] captured the basic simplicity of the problem by calculating the force between two conducting plates due to the modification which their presence imposes on the allowed (electromagnetic) modes. Cosmological implications are considered to be due to the finiteness of the universe [4]. The literature on the subject is vast, as is attested to in [4–8]. The vacuum for the electromagnetic field may be considered as its equilibrium state in the limit of vanishing temperature (T). It is then natural to study the extension of the above to finite temperatures. Indeed, several studies were published [9–12] where the Casimir free energy was obtained. Whence the force per unit area (pressure) on the plates could be calculated assuming that the Casimir free energy abides by the standard thermodynamic formulae.

In section 2 we obtain the Casimir energy at zero temperature from the high-temperature limit without recourse to regularization. This is accomplished by using what we call 'Kirchhoff's theorem'† (valid within classical physics) which states that the energy density, $u(T)$, of radiation confined in a cavity at equilibrium temperature T is a function of the temperature only at high temperatures. In section 3 we consider the expression for the Casimir free energy and relate it to the energy via the usual thermodynamic formula. We then define the Casimir entropy and display its temperature variation in a figure. The

† Kirchhoff's *Law* is a general relation (based on thermodynamics) between the radiative and absorptive power of a body held at a fixed temperature. The law implies that the total energy density is a function of temperature only ([5, p 2]). The latter is also implied by the equipartition theorem (e.g. [13])—hence we refer to the result that the energy density of the radiation at equilibrium at high temperature is a function of T only as 'Kirchhoff's theorem'.

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Casimir entropy, in the classical limit, is a *geometry-dependent* but *temperature-independent* constant (and, of course, its expression does not involve Planck's constant). In section 4 we use standard regularization techniques to show that at the high T limit, the Casimir energy and energy fluctuations (to all orders) vanish exponentially, in conformity with the classical 'Kirchhoff's theorem'. In section 5 we give a simple geometrical meaning to the Casimir entropy, note that at the high T limit the Casimir force (the plates' mutual attraction) is entirely entropic, and comment on possible future work.

2. The Casimir effect

The evaluation of the Casimir energy due to vacuum fluctuations for the case we consider is given in several texts and publications, e.g. [1, 4, 5]. We shall, therefore, present a sketchy outline of the set-up aimed primarily at fixing the notation. Thus, we consider the radiation confined between two conducting plates. The edge size of both plates is L. The first is placed at $z = 0$ in the XY plane. The second plate is placed at $z = a$ parallel to the XY plane. $L \gg a$; in fact we are interested in $L \to \infty$ while a remains finite. The energy tied down in the zero-point fluctuations of the fields in the presence of the plates is

$$E(a, T = 0) = \frac{\hbar c L^2}{2\pi} \int k_\parallel \, \mathrm{d}k_\parallel \left[\frac{k_\parallel}{2} + \sum_{m=1}^{\infty} k_m \right]$$
$$k_\parallel{}^2 = k_x{}^2 + k_y{}^2 \qquad k_m{}^2 = k_\parallel{}^2 + \frac{m^2\pi^2}{a^2} \qquad m = 0, 1, 2, \ldots. \tag{1}$$

The energy *density* in dimensionless units is

$$\varepsilon(a, 0) = \int x \, \mathrm{d}x \left[\frac{x}{2} + \sum_{m=1}^{\infty} \sqrt{x^2 + m^2} \right]$$
$$k = \frac{\pi}{a} x \tag{2}$$

while the dimensional energy density is given by

$$\frac{E(a, 0)}{L^2 a} = \frac{\hbar c}{2\pi^2} \frac{\pi^4}{a^4} \varepsilon(a, 0). \tag{3}$$

The zero-point fluctuations of the fields in the same volume without the constraining boundaries is, in our dimensionless units,

$$\varepsilon(\infty, 0) = \int x \, \mathrm{d}x \int \mathrm{d}m \, \sqrt{x^2 + m^2}. \tag{4}$$

The dimensionless Casimir energy density is

$$\varepsilon_C(0) = \varepsilon(a, 0) - \varepsilon(\infty, 0) = \int \mathrm{d}x \, x \left[\frac{x}{2} + \sum_{m=1}^{\infty} \sqrt{x^2 + m^2} - \int \mathrm{d}m \, \sqrt{x^2 + m^2} \right]. \tag{5}$$

Both $\varepsilon(a, 0)$ and $\varepsilon(\infty, 0)$ diverge. Thus, the evaluation of $\varepsilon_C(0)$ requires regularization. This is done most commonly [1, 8] via the introduction of a k-dependent function, $r(k/k_C)$, such that $r = 1$ for $k < k_C$ and $r = 0$ for $k \gg k_C \equiv 1/\alpha$. Such a cut-off represents the physics in as much as conductivity is a k-dependent quantity; for $1/k \ll d$—the interatomic distance—conductors become essentially transparent to radiation. For such waves the placement of the plates has no effect whatever. A convenient cut-off function is [1, 8]

$$r(\alpha k) = \exp(-\alpha k).$$

A detailed calculation will not be given here (cf, e.g. [1, 8]). The result for our case in our dimensionless units is

$$\varepsilon_C(0) = -\frac{2}{(2\pi)^4}\zeta(4) \tag{6}$$

where

$$\zeta(n) = \sum_{m=1}^{\infty} \frac{1}{m^n}.$$

We now turn to 'Kirchhoff's theorem'. The theorem states that the energy density of the radiation field enclosed in a cavity at equilibrium with temperature T is a function of T only. To see that the theorem is relevant to our case we consider a formal modification of the set-up specified above to the following one [8]. Consider six conducting plates forming a cube of edge length L. The corner of the cube is placed at the origin of our coordinate system. An additional (extra) conducting plate is placed at $z = a$, parallel to the XY plane. Thus, the cube is divided into two cavities of volumes L^2a and $L^2(L - a)$. The Casimir energy density is the difference in the energy densities of these two cavities. When the limit $L \to \infty$ is taken we arrive at the set-up we discussed above. Next we extend our analysis to the Casimir energy at finite temperature. This is most readily accomplished by including in the above expressions (equations (4) and (5)) the additional energy in each mode due to its thermal occupancy,

$$n(k, T) = \frac{1}{\exp(\frac{T_c}{T}x) - 1} \qquad k_B T_c = \hbar c \frac{\pi}{a}. \tag{7}$$

($k_B T_c$ serves as the characteristic energy; that this is so will become obvious as the calculations progress.) The expression for the Casimir thermal energy density is, then,

$$u_C(a, T) = u'(a, T) - u'(\infty, T) + u_C(a, 0). \tag{8}$$

Here $u'(a, T)$ $(u'(\infty, T))$ is the energy density of the constrained (unconstrained) system *without* the zero-point energy contributions (hence the prime on the u). Here by 'constrained' we mean with the conducting plates at $z = 0$ and $z = a$. $u_C(a, 0)$ is the Casimir energy density due to vacuum fluctuations,

$$u_C(a, 0) = \varepsilon_C(0)$$

of equation (5). (We use a different notation because here, in equation (8), $u_C(a, 0)$ is regarded as unknown, to be determined by means of 'Kirchhoff's theorem'.) $u'(a, T)$ is given by (after a trivial change of variables),

$$\begin{aligned} u'(a, T) &= D\left[f(0)/2 + \sum_{m=1}^{\infty} f(m)\right] \\ D &= \frac{\hbar c}{2\pi^2}\left(\frac{\pi}{a}\right)^4. \\ f(m) &= \int_m^{\infty} \mathrm{d}y\, y^2 n(y, T) \end{aligned} \tag{9}$$

while

$$u'(\infty, T) = D \int \mathrm{d}m\, f(m). \tag{10}$$

Now 'Kirchhoff's theorem' states that

$$u_C(T) \to 0 \qquad T \gg T_c. \tag{11}$$

Evaluation of the sum in (9) via the Poisson summation formula [8, 14] gives

$$u'(a,T) - u'(\infty,T) = D\sqrt{2\pi}\sum_{m=1}^{\infty} F(2\pi m) \tag{12}$$

where

$$F(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \mathrm{d}x\, \cos(\lambda x) f(x).$$

Direct integration yields ($t = \pi \frac{T}{T_c}$, $\lambda = 2\pi m$)

$$\Big[u'(a,T) - u'(\infty,T)\Big]/D = -2t^3 \sum_{m=1}^{\infty} \frac{1}{\lambda}\mathrm{ctnh}(t\lambda)\mathrm{csch}^2(t\lambda) + \frac{2}{(2\pi)^4}\zeta(4). \tag{13}$$

Using the results of equation (13) in equation (8) and noting that the sum involving the hyperbolic functions goes to zero (exponentially) as $t \to \infty$, we have from 'Kirchhoff's theorem' (11) our final result

$$u_C(a,0)/D = \varepsilon_C(0) = -\frac{2}{(2\pi)^4}\zeta(4). \tag{14}$$

Thus, we obtained the Casimir energy at $T = 0$ without recourse to regularization, but by integrations of the integrable temperature-dependent terms and assuming the validity of 'Kirchhoff's theorem' at high temperatures.

3. Casimir free energy and entropy

The Casimir free energy at finite temperatures has been calculated by several authors (e.g. [9–12]). Here also one calculates the difference between the free energy of the constrained system and the unconstrained one. Through an analysis quite similar to the one we considered above, in section 2, one obtains [10–12] ($\lambda = 2\pi m$)

$$f_C(T)/D \equiv \phi_C(t) = -t\sum_{m=1}^{\infty}\frac{1}{\lambda^3}\,[\mathrm{ctnh}(t\lambda) + (t\lambda)\mathrm{csch}^2(t\lambda)]. \tag{15}$$

Here $f_C(T)$ is the Casimir free energy per unit volume. The expression relates, as it should, to the Casimir energy density by the thermodynamic relation

$$\varepsilon_C(t) = \phi_C(t) + t\frac{\partial \phi_C}{\partial t}. \tag{16}$$

The zero-temperature limit yields $\phi_C(t \to 0) \to \varepsilon_C(0)$. Thus, it is natural to define Casimir entropy density, $\sigma_C(t)$, in units where $D = 1$, by

$$\phi_C(t) = \varepsilon_C(t) - t\sigma_C(t). \tag{17}$$

The temperature variations of these quantities are displayed in figure 1. The Casimir entropy density in our $D = 1$ units is, in the high T limit,

$$\sigma_C(t \gg 1) = \frac{\zeta(3)}{(2\pi)^3}.$$

Reverting to dimensional expressions, the Casimir entropy S_c at the high T limit is

$$\frac{S_c/k_B}{L^2 a} = \frac{\zeta(3)}{2^4\pi}\frac{1}{a^3}. \tag{18}$$

It is noteworthy that while the Casimir energy density vanishes in the high T limit (abiding thereby by 'Kirchhoff's theorem'), the Casimir free energy density does not. Thus, the resultant force of attraction between the plates is of entropic origin. We discuss this point further in section 5.

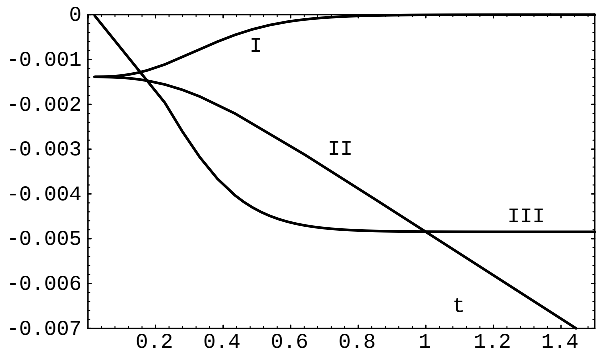

Figure 1. Casimir energy (I), free energy (II), and negative entropy (III) as a function of T. All quantities are in dimensionless units.

4. 'Kirchhoff's theorem' and Casimir energy fluctuations

In section 2 we used 'Kirchhoff's theorem' (cf footnote in section 1). It is of interest to give an explicit proof for the particular case under study here, namely, the Casimir energy density of the radiation field between two conducting parallel plates. This is done in this section.

The variation with temperature of the thermodynamic quantities considered in section 3 are seen to be essentially confined to $t \leqslant 1$. We will now consider the expression for the Casimir energy density (8) for $t \gg 1$. The analysis in this section requires regularization. The general formulae involved are given in detail in the appendix. The basic physical view of it is the existence, which we now assume, of a high frequency cut-off, ω_c, such that for temperatures $T \gg \hbar\omega_c/k_B$ we may expand occupancy terms appearing in the integrals as a power series in $(\hbar\omega)/(k_B T)$. Thus, the energy tied in a mode labelled by 'k' is

$$\hbar\omega_k[\tfrac{1}{2} + n(k, T)] \to k_B T. \tag{19}$$

This, of course, is recognized as the equipartition [13] result—each mode ties down $k_B T$ amount of energy—it being equivalent to a harmonic oscillator [5]. Now the evaluation of the thermal terms in (8) (namely, $u'(a, T)$ and $u'(\infty, T)$) requires regularization as both diverge. In the appendix it is shown that the final result is zero—i.e. the two terms cancel each other—they correspond to the $p = 0$ case (i.e. even). Thus, we have established 'Kirchhoff's theorem' for this case. Further, one may consider the Casimir energy fluctuations [11], i.e. the difference between the energy fluctuations in the constrained and unconstrained systems, at finite temperatures. Evaluation of the mean square energy fluctuation per unit volume gives

$$\Delta^2 E_C = D \int x\,\mathrm{d}x\, \frac{x^2}{2\left[(2\sinh(\frac{\pi x}{t})\right]^2} + \sum_{m=1}^{\infty} \int_m^{\infty} x\,\mathrm{d}x\, \frac{x^2}{\left[(2\sinh(\frac{\pi x}{t})\right]^2} - \int \mathrm{d}m \int_m^{\infty} x\,\mathrm{d}x\, \frac{x^2}{\left[(2\sinh(\frac{\pi x}{t})\right]^2}. \tag{20}$$

Upon taking the high T expansion we again have the case of p even (cf the appendix) which means the vanishing of the Casimir energy fluctuations at the high T limit. It can be shown that higher orders of the fluctuations also vanish in this limit.

The analysis can be viewed alternatively as follows. Given that the thermal occupancy is given by the Bose expression (7), we may calculate the Casimir thermal energy without the zero-point fluctuations contribution. Now at high T we have (cf equation (19)) as the energy tied to a mode,

$$\hbar\omega n(k, T) \to k_B T - \tfrac{1}{2}\hbar\omega_k.$$

Therefore, in order to have a vanishing Casimir energy (by 'Kirchhoff's theorem') we must add $\frac{1}{2}\hbar\omega_k$ to every mode—confirming thereby the form of the zero-point energy for the case of a Bose system.

5. Summary and conclusions

The analysis of the Casimir effect in a simple geometry as studied here involves essentially one characteristic parameter—the separation between the conducting plates, a. This parameter determines a characteristic temperature which we denoted by T_c. 'Kirchhoff's theorem', proven within classical physics, asserts that the energy density for the Casimir effect where one calculates differences in the energy density of cavities at equal temperature (T), should vanish. Using a form of the correspondence principle we took the theorem to hold for $T/T_c \gg 1$. This provides an equation that relates the contribution of the thermal energy to the temperature-independent Casimir energy density of zero temperature. This equation allowed us to calculate the zero-temperature Casimir energy from the high-temperature results for the thermal energy. The calculation gave the Casimir energy without recourse to regularization since at finite temperatures the integrals converge. The known result for the Casimir free energy (at finite T) was combined with that of the thermal energy to define, in a natural way, a Casimir entropy. It was shown to approach a constant value for $T/T_c \gg 1$ which is classical—it does not involve Planck's constant nor the velocity of light—and is temperature independent. It merely reflects the geometry of the problem. Thus, for the geometry considered, namely, the two parallel plates separated by a distance a, the entropy for $T/T_c \gg 1$ is essentially equal to the number of squares of edge a required to fill the area of the plate. From this vantage point the attractive force between the plates (the 'Casimir force') is entirely entropic—since by getting closer the number of such squares increases while the energy, as stressed above, is zero. It would be very useful here to derive this entropy from Boltzmann's method of counting the number of allowed states. It seems that the Casimir entropy is the simplest descriptive parameter for the Casimir effect—it being merely geometrical.

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Appendix. Regularization

Upon the introduction of the regularization function, $r(\alpha x)$, the general form of the expression that we wish to evaluate is

$$t_p = \int_0^\infty x\,\mathrm{d}x\,\frac{x^p}{2}\,f(x) + \sum_{m=1}^{\infty}\int_m^\infty x\,\mathrm{d}x\,x^p f(x) - \int_0^\infty \mathrm{d}m \int_m^\infty x\,\mathrm{d}x\,x^p f(x). \tag{A1}$$

(Equation (5) in section 2 corresponds to $p = 1$.) This equation leads to the following cut-off independent solution:

$$t_p = \lim_{\alpha\to 0}\frac{\mathrm{d}^{p+1}}{\mathrm{d}\alpha^{p+1}}\left[\frac{1}{2\alpha} + \frac{1}{\alpha^2}\left[\frac{\alpha}{\mathrm{e}^\alpha - 1}\right] - \frac{1}{\alpha^2}\right]. \tag{A2}$$

Using the well known [1, 8] expansion

$$\frac{x}{\mathrm{e}^x - 1} = 1 - \frac{x}{2} + \sum_{m=1}^{\infty}\frac{B_m x^{2m}}{(2m)!} \tag{A3}$$

(here B_m are the Bernoulli numbers [1, 8]) we see that for even p, $t_p \to 0$. For $p + 1 = 2m$, we obtain

$$t_p = \frac{B_{m+1}}{(2m)(2m-1)}. \tag{A4}$$

Thus, in evaluating the $T = 0$ case (i.e. equation (5)) we obtain

$$\varepsilon_C(0) = \tfrac{1}{720}.$$

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